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Self Consistent $1/N_c$ Expansion In The Presence Of Electroweak Interactions

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Abstract

In the conventional approach to the $1/N_c$ expansion, electroweak interactions are switched off and large N_c QCD is treated in isolation. We study the self-consistency of taking the large N_c limit in the presence of electroweak interaction. If the electroweak coupling constants are held constant, the large N_c counting rules are violated by processes involving internal photon or weak boson lines. Anomaly cancellations, however, fix the ratio of electric charges of different fermions. This allows a self-consistent way to scale down the electronic charge e in the large N_c limit and hence restoring the validity of the large N_c counting rules.

The $1/N_c$ expansion is now generally recognized as an invaluable tool in our handling of the non-perturbative nature of hadron dynamics. The pioneer work of 't Hooft [1] has proven that the large N_c limit is the weak coupling limit of meson dynamics. Quantitatively, a graph with k external meson legs can be *at most* of order $N_c^{1-k/2}$. For example, meson masses, described by graphs with two external meson legs, are of order N_c^0, while the "meson $\rightarrow$ meson + meson" decay amplitudes are suppressed by $N_c^{-1/2}$. As a result, mesons are stable and non-interacting in the large N_c limit. The generalization to include baryons was made by Witten [2], who had shown that a graph with two external baryon legs and k external meson legs are *at most* of order $N_c^{1-k/2}$. Hence the baryon masses ($k = 0$) and Yukawa couplings ($k = 1$) grow like N_c and $N_c^{1/2}$ respectively.

In the real world, however, hadrons experience not only QCD but also electroweak interactions. In the conventional approach to the $1/N_c$ expansion, one simply ignores the electroweak interactions and treats large N_c QCD in isolation. The results obtained for large N_c are then extrapolated back to $N_c = 3$ and applied to electroweak processes. This practice is permissible since electroweak coupling constants are independent parameters. It is interesting to ask what happens to the large N_c counting rules described above if the electroweak interactions are not switched off[1]. If the electroweak theory is not modified, it is easy to see that these counting rules will be violated by graphs involving electroweak currents.

One possible violation is the ρ meson two point function induced by ρ–γ mixing. As mentioned above, the ρ meson mass should be of order N_c^0. The ρ–γ mixing parameter, however, is just governed by the ρ meson decay constant f_ρ, which grows like $N_c^{1/2}$. It

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[1]We believe we are not the first ones to raise this question. In Chapter 7 of Ref. [3] Marshak made a cautionary remark about taking the large N_c limit "when the leptonic and quark sectors are both involved in the process" (pg. 450). We are, however, not aware of any systematic discussion on this topic in the literature.

follows that the contribution to the ρ mass by the ρ–γ–ρ mixing diagram grows like

$$m_\rho \sim f_\rho^2 \sim N_c^1, \tag{1}$$

violating the counting rule above. Another example is the $\pi^0 \to 2\gamma$ decay amplitude by the Adler–Bell–Jackiw anomaly [4,5],

$$A_{\pi^0} \sim N_c/f_\pi \sim N_c^{1/2}, \tag{2}$$

which diverges in the large N_c limit, in contradiction with the claim of meson stability made above. Moreover, such $\pi^0\gamma\gamma$ vertices can induced large $\pi^0\pi^0$ elastic scattering (through photon loops) amplitude of order N_c^2, violating the counting rule requirement that meson-meson elastic scattering amplitude should decrease like N_c^{-1}.

Such violations are also present in the baryon sector. Consider baryons with N_c quarks with the same flavor and hence the same electric charge. (For up and down quarks they are the large N_c generalizations of the Δ^{++} and Δ^- baryons respectively.) The electrostatic energies carried by such baryons grow like N_c^2, in violation of the counting rule that baryon masses should grow like N_c^1 only. These examples of violations of large N_c counting rules reflect the unsmoothness of the large N_c limit in the presence of electroweak interactions. Electromagnetic interactions introduce a correction of relative order $e^2 N_c$ to some strong processes. These effects vanish if we set $e^2 = 0$, but otherwise they diverge in the large N_c limit, independent of the particular values taken by e.

In this paper, we will try to show that the success of the $1/N_c$ expansion is no accident. There exists a well-behaved large N_c limit even in the presence of electroweak interactions. We will show that the ratio of electric charges carried by quarks and leptons are fixed by anomaly cancellation [6,7] of the underlying $\mathrm{SU}(N_c)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge theory. To achieve a smooth large N_c limit one can consistently scale down the electric charges carried by all the particles by a common power of N_c. This will introduce extra powers of $1/N_c$ to graphs involving photon currents and keep them in agreement with the large N_c counting rules. In addition, the modified electroweak interactions will remain a small perturbation to QCD, as they are in the real world.

For the $\mathrm{SU}(N_c)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge theory to be renormalizable, it is necessary to have all chiral anomalies cancelled. For example, the triangular anomalies [8], describing the interaction of three gauge bosons interaction through fermion loops, must be cancelled exactly. In one generation standard model, the fermions fall into the representations listed below:

Fields	$\mathrm{SU}(N_c)_c$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$
$\binom{u_L}{d_L}$	3	2	Y_Q
u_R	3	1	Y_u
d_R	3	1	Y_d
$\binom{\nu_L}{e_L}$	1	2	Y_L
e_R	1	1	Y_e

Only the following triangular anomalies do not cancel trivially and provide constraints on the hypercharges of different fermions.

$$\mathrm{U}(1)_Y^3: \qquad 2N_cY_Q^3 - N_cY_u^3 - N_cY_d^3 + 2Y_L^3 - Y_e^3 = 0, \tag{3a}$$

$$\mathrm{U}(1)_Y\mathrm{SU}(2)_L^2: \qquad N_cY_Q + Y_L = 0, \tag{3b}$$

$$\mathrm{U}(1)_Y\mathrm{SU}(N_c)_c^2: \qquad 2Y_Q - Y_u - Y_d = 0, \tag{3c}$$

and the mixed gauge-gravitational anomaly [9–11] provide a fourth constraint[2]:

$$\mathrm{U}(1)_Y(\mathrm{graviton})^2: \qquad 2N_cY_Q - N_cY_u - N_cY_d + 2Y_L - Y_e = 0. \tag{3d}$$

One can eliminate Y_L and Y_e from Eq. (3a) by Eq. (3b) and Eq. (3d). With $Y = \frac{1}{2}(Y_u - Y_d)$, Eq. (3c) gives $Y_u = Y_Q + Y$ and $Y_d = Y_Q - Y$, and Eq. (3a) becomes

$$2N_cY_Q^3 - N_c(Y_Q + Y)^3 - N_c(Y_Q - Y)^3 + 2(-N_cY_Q)^3 - (-2N_cY_Q)^3 = 0, \tag{4}$$

which can be further reduced to

[2]Yet another chiral anomaly, the global chiral SU(2) anomaly [12], constrain the number of left-handed fermion doublets to be even, hence requiring N_c to be odd and leaving the baryons as fermions.

$$Y_Q(N_c^2 Y_Q^2 - Y^2) = 0. \tag{5}$$

There are clearly two solutions to this equation. The "bizarre" solution [13] with $Y_Q = 0$ which gives

$$Y_Q = Y_L = Y_e = 0, \qquad Y_u = -Y_d, \tag{6}$$

is phenomenologically uninteresting for reasons detailed in Ref. [14]. That leaves us with the "standard" solution with $Y = N_c Y_Q$ (choosing $Y = -N_c Y_Q$ just reverses the labels "up" and "down" quarks),

$$(Y_Q, Y_u, Y_d, Y_L, Y_e) = (1, N_c + 1, -N_c + 1, -N_c, -2N_c)Y_Q. \tag{7}$$

All the hyperchrages are fixed up to an overall proportionality constant.

The electric charge is defined as,

$$Q = e(I_3 + Y/Y_0). \tag{8}$$

Since electromagnetic interactions conserves parity, the left-handed quarks and leptons must carry the same electric charges as their right-handed counterparts. This fixes $Y_0 = 2N_c Y_Q$ and

$$(Q_u, Q_d, Q_e, Q_\nu) = (\frac{N_c + 1}{2N_c}, \frac{-N_c + 1}{2N_c}, -1, 0)e. \tag{9}$$

By putting $N_c = 3$, the normal charge assignments are recovered. Hence we have shown that charge quantization follows from anomaly cancellations for arbitrary odd N_c. This observation is crucial for our later discussion as it provides a unique way to scale down all the charges of the quarks by scaling down the electronic charge e with anomaly cancellation all the way.

The world described by Eq. (9) shares many features of the real world. The neutrino is still electrically neutral, and the $Q_u - Q_d = -Q_e$ equality is preserved so that β-decays can still happen. In the large N_c limit, the up and down quarks carry charges $+e/2$ and $-e/2$ respectively ($2e/3$ and $-e/3$ in the real world), but the $q\bar{q}$ mesons still have charges e, 0, or

$-e$ as in the real world. The proton has $(N_c+1)/2$ up quarks and $(N_c-1)/2$ down quarks and hence its charge is

$$Q_p = e\left(\frac{N_c+1}{2}\frac{N_c+1}{2N_c} - \frac{N_c-1}{2}\frac{N_c-1}{2N_c}\right) = e, \tag{10}$$

and the hydrogen atom stays neutral. The neutron, on the other hand, carries no electric charge as usual.

$$Q_n = e\left(\frac{N_c-1}{2}\frac{N_c+1}{2N_c} - \frac{N_c+1}{2}\frac{N_c-1}{2N_c}\right) = 0. \tag{11}$$

Coming back to large N_c counting rules, the $\pi^0 \to 2\gamma$ decay is given by,

$$A_\pi^0 \sim \frac{N_c(Q_u^2 - Q_d^2)}{f_\pi}. \tag{12}$$

As mentioned above, $N_c/f_\pi \sim N_c^{1/2}$ but now we have an additional suppression factor from the electric charges, $Q_u^2 - Q_d^2 = e^2/N_c$. Hence

$$A_\pi^0 \sim \frac{e^2}{f_\pi} \sim N_c^{1/2}, \tag{13}$$

and the counting rules are satisfied.

The ρ–γ mixing problem, however, still persists. The ρ–γ mixing amplitude $A_{\rho\gamma}$ is given by,

$$A_{\rho\gamma} = f_\rho(Q_u - Q_d) = e f_\rho, \tag{14}$$

which diverges as before. Also, the Δ baryon self-energy still diverges as N_c^2, violating the counting rules.

As suggested before, one of the possible remedies to the situation is to scale down the electronic charge e in the large N_c limit, providing extra suppression factors. Since we are scaling the strong coupling constant g_3 by keeping $g_3^2 N_c =$ constant, it is natural to impose the electric charge scaling condition as

$$e^2 N_c = \text{ constant}, \qquad \text{as } N_c \to \infty. \tag{15}$$

With $e^2 \sim N_c^{-1}$, $A_{\rho\gamma}$ is suppressed in the large N_c limit,

$$A_{\rho\gamma} = e f_\rho \sim N_c^0, \tag{16}$$

and the Δ baryon electrostatic self energy is

$$M_{elec} \sim e^2 N_c^2 \sim N_c, \tag{17}$$

exactly as specified by the counting rules. In general, it is easy to prove that Eq. (15) is sufficient to keep all the large N_c counting rules intact even in the presence of photons. We first note that the $q\bar{q}\gamma$ vertex is of the same order as the $q\bar{q}g$ vertex (both of order $N_c^{-1/2}$), and replacing an internal gluon line from a planar diagram with a photon line does not produce additional powers of N_c. An analysis similar to the one given by Witten [2] can be readily carried out. Moreover, the couplings of quarks to leptons or $W^\pm$ via exchange of photons present no difficulties as a consequence of Eq. (15). Thus the graphs with photon lines are either of the same order in N_c as the leading planar diagrams, or are simply dominated by the latter. Hence condition (15) is sufficient to guarantee the validity of the large N_c counting rules.

Our conclusions can be easily generalized to the case of weak currents. The graphs with $W^\pm$ and Z^0 lines can violate the large N_c counting rules unless the conditions like

$$g_2^2 N_c = \text{ constant}, \qquad \text{as } N_c \to \infty, \tag{18}$$

are imposed, where g_2 is the $\mathrm{SU}(2)_L$ coupling constant. It is a general feature that *all* coupling constants must be scaled down correspondingly even though we are taking the large N_c limit of only *one* of the gauge groups.

One should also note that the large N_c scaling conditions Eq. (15) and (18) are not the only ones which lead to a smooth large N_c limit. It is easy to see that any scaling conditions like

$$e^2 \sim N_c^{-m}, \qquad \text{as } N_c \to \infty, \tag{19}$$

with $m \geq 1$ is going to give a smooth $1/N_c$ limit. Conditions (15) and (18) are just the critical cases with $m = 1$. A large suppression power m, on the other hand, will lead to severe

suppression of electroweak effects in the large N_c limit. It is noted that the conventional approach of switching off the electroweak interaction before taking the large N_c limit is equivalent to taking $m \to \infty$ in this formalism.

Lastly, it is natural to ask if there exists any well-defined limit of the weak mixing angle θ_W in the large N_c limit. It seems to us that, since the coupling constants of $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$ are independent quantities, the value of θ_W is not constrained unless we start with some grand unified gauge group. It turns out that no simple analogs of SU(5) or SO(10) grand unifications exist for $\mathrm{SU}(N_c)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Hence θ_W is unconstrained in the present stage of our understanding of the large N_c limit.

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